Extra Problems on Vector Calculus

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October 28, 2009

Note: Problems marked with $\star$ are more difficult. Try the easier ones first.

1. Write in spherical coordinates: $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1 \text{ and } x, y, z \leq 0\}$. Integrate x over this region.
2. The plane $y = z$ slices the cylinder $x^2 + y^2 = 4$. Parametrize the resulting curve.
3. Let S be the unit sphere centered at the origin, with unit normal directed outward. Set $\vec{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the flux of $\vec{F}$ across S in three ways:
   (a) Directly.
   (b) Geometrically.
   (c) Using the Divergence Theorem.
4. Write in cylindrical coordinates: $\{(x, y, z) : x^2 + y^2 \leq 1 \text{ and } x^2 + z^2 \leq 1\}$. Find the volume of the solid.
5. Evaluate $\displaystyle\iiint\limits_{\mathbb{R}^3} \frac{dV}{\left[1 + (x^2 + y^2 + z^2)^{3/2}\right]^{3/2}}$.
6. The electrostatic field is a vector field $\vec{E}$ which, among other things, satisfies $\nabla \times \vec{E} = 0$ everywhere. Given a closed curve C is space, evaluate the line integral of $\vec{E}$ along C either using the fundamental theorem of line integrals or Stokes's theorem.
7. A uniform fluid flow is directed vertically (say, heavy rain) and is given by the vector field $\vec{F} = (0, 0, -1)$. Find the flux through the cone given by $z = \sqrt{x^2 + y^2}$, where $0 \leq z \leq 1$ and the normal is directed downward.
8. Let R be the volume in space which inside the cylinder $x^2 + y^2 = 4$, outside the cylinder $x^2 + y^2 = 1$, and between the planes $z = 0$ and $z = 4$. Set $\vec{F} = (x + yz, y + x^2z^3, xyz)$ and compute the flux out of R.
9. Symmetry arguments: Let B be the unit ball centered at the origin in space and S denote the top ($z \geq 0$) half of the boundary, oriented upwards.
   (a) What is the flux of $\mathbf{i} + 2\mathbf{j}$ across S?
   (b) Integrate $f(x, y, z) = xy$ over B.
   (c) Integrate $f(x, y, z) = 2x + 3$ over S.
10. $\star$ Find the average distance of an arbitrary point of the unit ball (in $\mathbb{R}^3$) to the north pole.
11. $\star$ Find the volume of the region which is inside all three cylinders: $x^2+y^2 \leq 1$, $x^2+z^2 \leq 1$, and $y^2+z^2 \leq 1$.
12. $\star$ For every product rule of derivatives there is a corresponding integration by parts. Let $R > 0$ and B be the ball of radius R centered at the origin in $\mathbb{R}^3$.
   (a) Begin with $\nabla \cdot (f\vec{F}) = (\nabla f) \cdot \vec{F} + f\nabla \cdot \vec{F}$ and show
   $$\iiint_B f(\nabla \cdot \vec{F})\, dV = \iint_{\partial B} f\vec{F} \cdot d\vec{S} - \iiint_B (\nabla f) \cdot \vec{F}\, dV.$$
   (b) Let $\vec{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Evaluate
   $$\iiint_B e^{-\rho}\, \nabla \cdot \left(\frac{\vec{\rho}}{\rho^3}\right)\, dV.$$
   The integral cannot be done directly without knowledge of the Dirac δ 'function'.